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THE OXIDATION OF HYDROQUINOL IN THE PRESENCE OF ALIPHATIC AMINES

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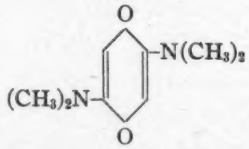
That hydroquinol is converted by oxidation in neutral and acid solutions into quinone, with the intermediate formation of quinhydrone, was first demonstrated by Wöhler in 1844.[2] This reaction appears to proceed quantitatively and the change can be brought about by the use of a great variety of oxidizing agents.

When we turn, however, to the oxidation of hydroquinol in alkaline solution the situation is far more complicated and at present we must admit that we have not an accurate understanding of the various changes which take place. The writer's interest in this phase of the subject resulted from the observation that an aqueous solution of hydroquinol, made strongly alkaline with methylamine, rapidly absorbs oxygen gas at ordinary temperatures with formation of a brilliant purple colored compound which is characterized by its pronounced crystalline habit.

It is of course well known that alkaline solutions of hydroquinol absorb oxygen, but if one will substitute an aliphatic amine, like methylamine, for the alkali usually employed he will observe an absorption of oxygen which is unusual both for its rapidity and in regard to the quantity consumed. The characteristic reactions which take place under these conditions appear to have not been studied.

For example, if hydroquinol is dissolved in a strong aqueous solution of mono- or dimethylamine and this solution is placed in a tube and exposed to oxygen, the liquid rapidly assumes a brownish red color and within an hour the deposition of a considerable quantity of beautiful red plates is observed. At the end of twenty-four hours the solution is almost filled with these brilliant crystals and for each gram of hydroquinol used there is absorbed about 250 cc. of oxygen. If the oxidation is continued for two or three days more the total absorption will increase to 500–600 cc. of oxygen per gram of hydroquinol, or from 10 to 12 equivalents of oxygen for

each molecule of hydroquinol. The colored crystalline compounds formed under such conditions are very stable and can be easily purified. The compound formed by the oxidation of hydroquinol in the presence of dimethylamine melts at 171° and is to be expressed by the formula $C_{10}H_{14}O_2N_2$. While its constitution has not yet been completely established it appears to be identical with tetramethyl diamino-quinone which was obtained by Mylius[3] by treatment of benzo-quinone with dimethylamine, and to which Kehrmann[4] has assigned the structural formula:



Aliphatic derivatives of this type are analogous in constitution to dianilino-quinone which Hofmann[5] prepared by interaction of aniline with quinone. While a large number of these anilino-quinones have been synthesized, an examination of the literature has shown that only four other representatives of these aliphatic amino quinones are known: the methylamino derivatives of toluquinone,[6] 2-5 xyloquinone,[7] and thymoquinone,[8] and the ethylamino derivative of thymoquinone.[6] These were prepared by interaction of the amine and quinone with yields which were said to be very low. The above mentioned oxidation of hydroquinol appears, therefore, of special interest as a simple, efficient means of preparing these aliphatic amino quinones, which latter compounds seem to present a chemical field as yet very little developed.

A somewhat analogous oxidation in the naphthalene series has been recorded by Plimpton.[9] He prepared amino derivatives of naphthoquinone by the action of amines upon naphthoquinone and notes that these compounds are also produced when a solution of naphthoquinol and the amine is exposed to the air. The resulting product, however, was a *mono*-aminoquinone and not a *di*-aminoquinone such as we appear to have with the benzene series.

The brilliant purple compounds corresponding to formula I are not the only products resulting from this oxidation of hydroquinol and aliphatic amines. The yield of these purple compounds never approaches quantitative values, and on concentrating the filtrate to expel the excess of free amine there is deposited, upon cooling, a considerable amount of dark yellow crystals indicating at least another compound. This latter is now being investigated. It has also been found that when an aliphatic amine is added to hydroquinol the first product of reaction is a double compound or addition product of the amine and hydroquinol. Several such com-

binations have already been prepared and isolated. Whether they form an intermediate step in the process of oxidation cannot be stated at present with certainty. It has also been observed that the oxidation proceeds rapidly even in air, and quite as well in alcohol as in water.

Research is now in progress dealing with the oxidation of hydroquinol, and its homologues toluquinol, xyloquinol, etc., in the presence of various aliphatic amines. The detailed results of this work will be published later in the *Journal of the American Chemical Society.*

[1] NATIONAL RESEARCH FELLOW IN CHEMISTRY.
[2] Wöhler, *Ann.*, **51**, 1844 (145–63).
[3] F. Mylius, *Ber.*, **18**, 1885 (467).
[4] F. Kehrmann, *Ibid.*, **23**, 1890 (906).
[5] A. Hofmann, *Jahr.*, 1863 (415).
[6] F. Fichter, *Ann.*, **361**, 1908 (397, 400).
[7] F. Fichter, and A. Willmann, *Ber.*, **37**, 1904 (2388).
[8] Th. Zincke, *Ibid.*, **14**, 1881 (94).
[9] R. T. Plimpton, *J. Chem. Soc.*, **37**, 1880 (637).

THE GRATING SPACE OF MICA AND THE INTENSITIES OF THE SPECTRAL ORDERS

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In attempting to determine the reflectivity from its principal cleavage planes, it was found desirable to know the grating space and incidentally the relative intensities of the several spectral orders.

We would like to point out that perhaps the grating space of mica may not be constant. The thickness of a piece of mica may be altered by pressure to a small extent without apparently destroying its crystalline structure. In this case we should expect nautral specimens even of the same chemical constitution to show differences due to the variable pressures to which they may have been subjected in their natural positions in the native rocks.

The specimen of mica here reported on was a clear sheet about one millimeter in thickness. It was selected for its smoothness of surface and freedom from flaws. Nevertheless it was not optically plane as shown by viewing a distant straight edge by reflection from the surface. Small local departures from a plane surface were evident.

The measurements were made by an ionization spectrometer, the standards of wave-lengths were the $K_{\alpha_1}(\lambda = .710)$ and the $K_{\beta_1}(\lambda = .633)$ emitted from a Coolidge tube having a molybdenum target. The slit of the ionization chamber was left open wide, the readings of angular position being made only on the crystal verniers.

Measurements were made on the whole spectrum, line and continuous. The relative heights of the peaks of the curves for the two lines after proper correction for the natural leak and the continuous spectrum was taken as the relative intensities of the lines in the several orders. Readings were carefully made on each side, left (L) and right (R) of the zero position of the crystal. The expression $^1/_2(L - R - 180°)$ is then the mean grazing angle θ for the particular line and order.

The results are given in the table. The spectra of certain orders were so weak and the curve so flat that it was not possible to fix the position of the peak with sufficient accuracy to be of value in the calculation of the grating space.

The relative intensities of the α_1 and β_1 lines are given in the table. The odd orders are stronger than the even orders as observed by de Broglie. The third and fifth orders are stronger than the first. Radiation at higher orders than the 7th was observable.

The grating space d calculated for the two lines is given in the table. The values agree fairly well with those calculated by Seigbahn (*Phys. Rev.*, Sept., 1916) from the measurements made by Gorton (*Phys. Rev.*, March, 1916), for certain lines in the L radiation of tungsten.

It is noticeable that the grating space as here determined increased with increasing spectral order. The results calculated by Seigbahn from Gorton's measurements show the same tendency. This effect has also been noted by W. Stenström (Untersuchengen der Röntgen Spektra, Lund, 1919). Stenström attributes this progressive change in the apparent value of d to a very small refraction of the X-rays in the crystal. This is a possible explanation of the effect but not the only one. It may well be due to small curvature or irregularities in the crystal. One can readily show that if the reflecting planes do not line accurately at the center of rotation of the crystal, or if the beam of X-rays from the slits do not pass accurately across this center of rotation, there will be a progressive change in the calculated values of d for the various orders if the crystal planes have even a small curvature. The direction of the change of d with order will depend on whether the crystal is convex or concave.

TABLE 1

	β_1 LINE		α_1 LINE		INTENSITY	
N	θ	d	θ	d	β_1	α_1
1	1°-50′-50″	9.87	2°- 4′-0″	9.845	100	364
2	3-39-38	9.914	4- 7-7.5	9.885	68	221
3	5-3-30	9.891	6-10-15	9.901	154	544
4			8-13-15	9.931	45	175
5	9-8-38	9.957	10-16-38	9.949	136	508
6					9.1	28
7			14-27-0	9.958	32	101

IONIZATION AND RADIATION POTENTIALS AND THE SIZE OF THE ATOM

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Professor A. S. Eve has recently published an interesting note on a relation between the ionization potential and the size of the atom. (*Nature*, June 30, 1921).

For some time I had been accumulating data for a similar comparison, but from a somewhat different point of view. Eve considers that the size of the atom is determined by the radius of the outer ring. The work (ionization potential) required to lift an electron from this ring by the Bohr model is proportional to $1/a$. The product Ixa should be a constant. Trials of this relation are made for a number of elements using both the cube-roots of the atomic volumes and estimates of the diameters of atoms made by W. L. Bragg from crystal structure and measurements. (*Phil. Mag.*, Aug., 1920). The results obtained by Eve showing the degree of constancy (and departures from constancy)are repeated here in columns 6 and 7 of the table.

The boundary of the atoms, however, should not correspond exactly with the outer ring, but should extend beyond it. If the electron ring were the limit in the solid state, it would require no work to remove an electron from an atom in this state. The photo-electric effect shows that work is required, that it is less than the ionizing potential, but is approximately the same as the radiation potential. I here consider that the limit of the atom is the ring or distance from the center that an electron needs to be lifted to produce the radiation potential. That is, an atom in the solid state of matter is ionized at a potential about equal to the radiation potentials in the gaseous state. The ionizing potential is given by

$$I = A/a,$$

where A is a constant and a is the radius of the electron ring or orbit. If b is the distance from the atomic center then an electron must be lifted to produce radiation, then the radiation potential is given by

$$R = A/a - A/b,$$

where A/b is the work required to remove an electron from position b entirely from the atom in the gaseous state.

$$I - R = A/b,$$

Th product $(I - R)b$ or, for comparison with atomic volumes, $(I - R)\sqrt[3]{A.V.}$ should be a constant. The last two columns of the table show

the results obtained for the above products. The values of the atomic diameters expressed in Ångstrom units are repeated from Eve's paper which were taken from a list by Bragg.

TABLE 1

E	I	R	D	$\sqrt[3]{A.V.}$	I × D	I × $\sqrt[3]{A.V.}$	(I−R) × D	(I−R) $\sqrt[3]{A.V.}$
H	13.8	10.2		2.45		33.8		8.82
				Group I				
Na	5.13	2.12	3.58	2.87	18.1	14.7	10.74	8.61
K	4.32	1.55	4.15	3.57	17.9	15.4	11.5	9.64
Rb	4.15	1.55	4.50	3.81	18.7	15.9	11.7	9.90
Cs	3.9	1.48	4.75	4.12	18.4	16.0	11.5	9.97
				Range	.8	1.3	1.00	1.37
				Group II A				
Mg	7.61	2.65	2.85	2.4	21.7	18.3	14.14	12.24
Ca	6.09	1.90	3.40	2.96	20.8	18.0	14.28	12.16
Sr	5.67	1.80	3.90	3.25	22.2	18.4	15.2	12.57
Ba	.19	1.56	4.20	3.31	21.8	17.2	15.25	12.03
				Range	1.4	1.2	1.11	.54
				Group II B				
Zn	9.35	4.02	2.65	2.09	24.77	19.5	14.12	11.2
Cd	8.95	3.79	3.20	2.35	28.64	21.0	16.25	12.12
Hg	10.38	4.85		2.45		25.4		13.47
				Range	3.87	5.9	2.13	2.27
				Group III B				
Tl	7.3	1.07	4.50	2.58	32.85	18.8	28.	16.07
				Group IV B				
Pb	7.93	1.26	3.80	2.63	30.13	20.8	25.38	17.54
				Group V A				
As	11.5	4.7	2.52	2.36	29.	27.2	17.14	16.00
				2.52		29.0		17.1
P	13.3	5.8		2.37		31.5		17.78
				2.57		34.2		19.27
				Range		4.3		1.78
						5.2		2.17
				Group VI				
S	12.2	4.78	2.05	2.5	25.	30.4	15.17	18.5
				Group VII A				
I	10.1	2.34	2.8	2.95	28.3	29.8	21.73	22.9
				Inert Gases				
He	25.6	20.5		2.86		73.		14.25
Ne	21.3	16.9	1.3	2.67	27.7	42.8	5.72	13.33
A	15.5	11.5	2.05	3.03	31.77	36.4	8.2	12.1
				Range	4.00	36.6	2.5	2.15

The values of the radiation potentials for strontium and barium have not been directly determined, but Mohler, Foote and Megers (*Sci. Paper, Bureau of Standards*, No. 403, 1920) give estimates from the wave-lengths of the tail lines of the proper spectral series for these elements. Hydrogen is added to the table, although not given by Eve. An inspection of the table brings out the following:

For Group I, the degree of constancy of the products is nearly the same, being slightly in favor of the use of I rather than (I — R). Group II A, shows, on the other hand, that (I — R) gives a better agreement than I. This is still more marked in Group II B. Group V A is considerably more favorable to (I — R). In case of the inert gases the product using I alone does not hold at all, while the product (I — R) gives nearly a constant. The product using (I — R) places hydrogen in good agreement with the first Group. Mercury, which has some of the physical qualities of the inert gases, has a product about equal to that for those gases. In case of the whole table, going from Group to Group, the products using (I — R) are much more constant than is the case with I alone.

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*PERIODS AND LOGARITHMIC DECREMENT OF THE GRAVITATION NEEDLE UNDER HIGH EXHAUSTION**

BY C. BARUS

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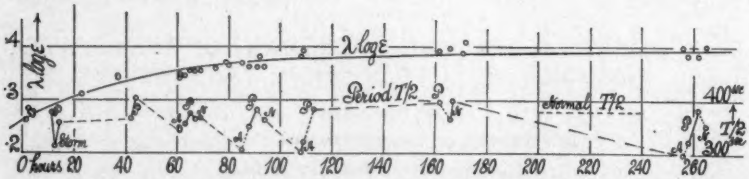
The Deflections.—After a search for a finer quartz fibre than was used in my last paper, one was found giving a double deflection of $\Delta y = 13.42 \pm .03$ cm. as compared with the former 2.67 cm., for the same case and needle. In spite of this astonishing sensitivity and otherwise admirable behavior of the apparatus, the new result for Δy from 30 successive night observations came out relatively less accurate than the former. At the same time, the observations on any single night (details must be omitted here) rarely differed by more than .1%. It is a case therefore in which the increasing importance of the radiation forces renders further finessing with the fibre of doubtful use in the given environment.

Logarithmic Decrements λ log ϵ.—In the endeavor to cope with this formidable difficulty, I began a study of the vibration of the needle in high vacua of a few thousandths of a millimeter, but the summer vanished before I completed it. The results so far obtained are interesting, however, and are given in the attached chart. The case was exhausted in the beginning to about 2.6×10^{-3} mm. on the McLeod gauge, and then sealed off. Air, however, in the lapse of 11 days very slowly leaked in through the glass cock and rubber tubes; and as the installment prevented me from finding the successive vacua (always beyond the U-gauge limit) I have expressed

the results for T and λ in the lapse of time. To determine the latter, the needle was deflected from rest to the maximum elongation, by placing the lead balls M M' in either of the active positions; and with the needle so deflected for an instant, to quickly put the weights in the neutral position again; i.e., in the vertical plane through the needle. It is thus possible to determine both the period T of the needle and the logarithmic decrement: the former by timing successive passages (first and second) through the equilibrium position with a stopwatch, and the latter by reading the elongations. The zero point was nearly constant. The radiation is thus taken symmetrically from both sides. So far as the method goes, the values of T and λ are quite accurately found in this way, the difficulty being in the interpretation. The plenum value of $\lambda \log \epsilon$, so far as determinable in a heated room was about 1, the needle being now practically aperiodic. This has decreased to about .27 (ratio of arcs 1.9) at 2.6×10^{-3} mm. of pressure and the curve suggests a more rapid decrease beyond. With the pressure values added the curve will afford data for the viscosity of highly rarified air.



Periods.—Unlike λ, the values for the period T (figure) show no marked improvement as the vacuum increases. In spite of the steady zero point, they rather vary with the time of day, being smallest in the morning and largest toward the end of the day and they drop off at night (A denoting forenoon, P afternoon, N night). The latitude of values is enormous, ranging from about 10 minutes to over 13 minutes. The former cases (low T) are usually encountered in the morning, and probably indicate repulsive radiant forces acting from the plates toward the middle plane of the case, like an elastic buffer, whereby the period is shortened. Moreover the effect of the changes of λ on T are nearly negligible in comparison with the enormous fluctuations of T resulting from modified environmental radiation.

Whereas the deflections Δy are largest in the morning and least at night, the T values pursue a nearly opposite course, increasing from morning to afternoon, except that the night periods again drop off. When the environmental temperature is gradually increasing (morning) and when it is gradually decreasing (night), small values of T, the period of the needle, occur. This was also shown strikingly during a violent rain storm (chart 11–15 hours), in which the atmospheric temperature dropped about 20° F.

If the case is warmer, the needle meets repulsive agencies toward the center from the excess of radiation inward. If the masses m at the end of the needle are the warmer, the needle radiates toward the case outward and therefore experiences an equivalent repulsive reaction. Finally with a non-uniform radiant field of force, increasing from needle to case, the effect will be virtually an increase of the torsion coefficient, of the quartz fibre, so that T comes out smaller.

It would then follow that the large deflections are obtained when the radiant field is nearly uniform and that the larger values of the period T are the more nearly correct. This finally is in keeping with the γ-values thus obtained.

Finally the deflections Δy are modified and complicated by the screening effect of the presence of the masses M, which in the T experiments are absent.

Estimate of Radiation Temperature-differences.—Let us suppose that the radiant repulsion from case to needle, the elastic buffer effect explained, can be expressed as an increment a_r of the modulus of torsion a, of the needle. Furthermore, if f_r is the radiant force at each end and $\Delta y_r/4L$ half the double deflection in radians, we get a second value of a_r. If we equate the two values

$$f_r = \pi^2(ml/L)\,\Delta T^{-2}.\,\Delta y_r \qquad (1)$$

To find $\Delta 1/T^2$, the least and maximum T found in vacuo in the last paragraph, i.e., $T' = 5$ minutes and $T = 6.5$ minutes, may be inserted, so that the modulus in (1) is 7.0×10^{-4} nearly.

To determine the radiation forces in vacuo, it will be necessary to postulate the case of black body, or full radiation. If θ and θ_0 are the absolute temperatures between which radiation takes place and E the energy radiated (ergs per cm.^{2} per sec.)

$$E = 4K\theta^3\Delta\theta$$

The value θ may be taken as 300° and $K = 5.3 \times 10^{-4}$. Thus

$$E = 10^3 \times 5.7\,\Delta\theta.$$

This radiation may at the outset be supposed to reach the needle from a single cm.^{2} S, S', of *each* plate situated on the end normal of the needle and plate. Thus eventually, $\rho_0 = E/2\pi c$ (c being the velocity of light) is the energy density or the light pressure at a distance of 1 cm. from the element S or S'. If we suppose (as is nearly the case) the shot m of frontal area A to be situated there, the force f_r upon it will be $\rho_0 A$, where $A = .16$ cm.^{2} Hence

$$f_r = 10^{-9} \times 4.8\,\Delta\theta \qquad (2)$$

for the square centimeter of plate taken.

If now we equate the values of f_r in (1) and (2)

$$\Delta\theta = 150\,\Delta y_r \text{ nearly,} \qquad (3)$$

if but a single square cm. on each glass face radiates toward m. As there are a number of such square cms., contributing radiation, the coefficient of (3) would be decreased ten or twenty times. One may conclude I think that $\Delta\theta$ compared with Δy_r is numerically much in excess. This implies too large a $\Delta\theta$, to persist in a room of practically constant temperature, in the summer. It is thus more than probable that at a vacuum of 10^{-3} mm. or a force of .22 dynes on each shot, so small a part as 10 or 20 times the $10^{-3} \times 70 \times \Delta y_r$ instanced in equation (1) may remain fluctuating in value with small changes in the radiating thermal environment.

Measurement of γ in Terms of the Viscosity of Air.—A maximum deflection of the m-balls is produced by placing the M-balls in position. When this is attained, the latter are reversed and the velocity v with which the needle (filamentary frame) passes through the equilibrium position, is measured with a stopwatch. The M-balls are then again reversed and the measurement of v repeated; etc. In a plenum, the motion of the needle through the zero point is practically uniform and therefore the frictional force is equal to the gravitational pull. Moreover, if two points of the scale near to and equidistant from the zero point are selected for releasing and arresting the stopwatch, the torsion of the fibre acts to the same degree as an acceleration and a retardation.

The frictional resistance encountered by either round mass m of radius r is by Stokes' equation $6\pi\eta rv$, where η is the viscosity of air. The velocity v of the balls m is

$$v = (l/2L)\,\Delta y/\Delta t$$

Δy is the telescopic excursion in the time Δt, symmetrically, to the position of equilibrium. We obtain in this way

$$\gamma = \frac{R^2}{Mm}\,\frac{3\pi\eta rl}{L}\,\frac{\Delta y}{\Delta t}$$

Work on this principle has been actively pursued, but the summer closed before it could be completed.

* Advance note from a Report to the Carnegie Institution of Washington, D. C.

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*PLANE REFLECTION OF SOUND, AS EXHIBITED BY THE PIN-HOLE RESONATOR**

BY C. BARUS

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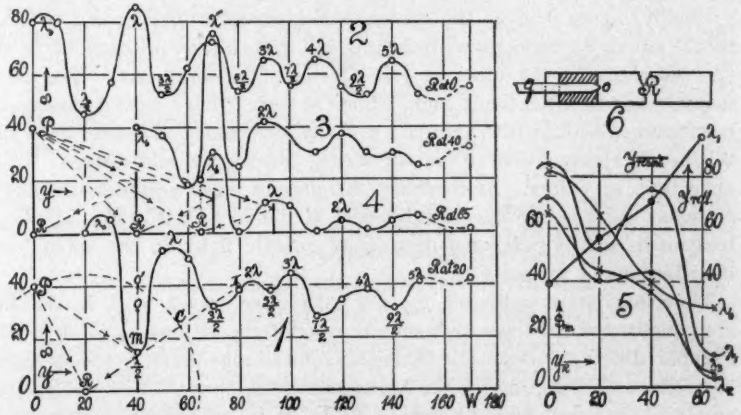
1. *Vertical Reflectors.*—These experiments, made between the walls of a room (y direction), with the closed organ pipe P, vertical, mouth down-

ward, at $z = 40$ cm. above the origin of the table, came out with surprising definiteness at once. They will therefore aid in disentangling the mysterious behavior of the resonator elsewhere observed (cf. *Science*). The resonator R, fig. 6, pin-hole tube qo, was placed on the table, horizontally, with the mouth toward the pipe (azimuth 270°). The vertical board or reflector 4 feet $\times$1 foot placed normal to y, was then moved successively from $y =$ 20 cm. (about at the mouth of R) nearly to the wall W at $y = 174$ cm. The results are given in figure 1 where s is the fringe deflection of the U-tube interferometer. The chief maxima are near $y = 50$, 100, 150 cm., which suggests the wave-length of the f″ pipe $\lambda = 48$ cm., here operating by oblique reflection. Between the maxima λ and 3λ, there is a subsidiary maximum which one might erroneously attribute to overtones, inasmuch as it is often associated with weakly or strongly blown pipe notes. Thus at $y = 80$ the largest deflection is obtained from the dying note,



at 90 from a forced note; but the 2λ crest is regular. Distances between 150 cm. and the wall W at 174 cm., were not available. A remarkable feature of the curve is the strong initial minimum at $y = 40$ cm. If the board is placed more or less obliquely to y, the minimum tends to vanish, as shown by the successive tests at o' and o.

In the disposition given, two sound rays reach the resonator, one reflected from the board and the other direct, with immediate reflection from the table. When the board is near the mouth of the resonator, these rays are reduced to one and the result is an accentuated maximum. As the board distance in y increases, the path difference steadily increases with a simultaneous weakening of the reflected ray; nevertheless the figure proves conclusively that it is not as ineffective as one might suppose.

In contrast with the preceding, the next graph, figure 2, for the same pipe position, but with the resonator on the table at $y = 0$ cm. (in azimuth

180°) is more regular. The curve lies in higher values of s. There is less repetition of sectional groups. Nevertheless the same number of six maxima occur; i.e., each max. in figure 1 is reproduced and often accentuated in figure 2. The former minimum of $y = 40$ has shifted to a minimum somewhat above 20 cm.

The third survey in y was made with the pipe again at $z = 40$ cm. above the origin and with the resonator on the table (in azimuth 180°) at $y = 40$ cm. The graph, fig. 3, for the plate reflector normal to y and advancing along it shows the same number of maxima as heretofore, with the sixth soon to vanish. The feature of the curve is the enormously developed second maximum, in contrast with the first, or the others. We are no doubt trenching on the case of favorable incidence heretofore developed. The response was always stronger to the weak notes and they were used throughout, the difference being often over 10 scale parts, s.

Finally, figure 4 gives the corresponding acoustic distribution evoked by the advancing reflector with the pipe left in its former position, but with the resonator on the table at $y = 65$ cm. in azimuth 180°. The first maximum is now relatively high, while the former huge second maximum has flattened and the fifth and sixth maxima have vanished behind the wall. Weak notes were essential, as the strong notes gave relatively as little deflection, as before. As a whole, the present work adduces strikingly clear-cut evidence of the interferences to be identified in the following treatment; but it fails, as yet, to shed definite light on the anomalous distributions of intensity, s.

These maximum ordinates, s_m, for path-differences $\lambda = 0, \lambda, 2\lambda, 3\lambda$, are constructed for the corresponding resonator positions, y_r in figure 5. Compared with the regularly diminishing initial maximum, λ_0 the strongly harmonic s–distribution for the λ maximum, together with its decreasing amplitude are well brought out. Like λ_0, 2λ is but sparingly harmonic, while the odd 3λ is again markedly so. The graphs, fig 5, also contain the relation of the reflector and resonator positions in y, for λ_1 crests. The broken line should probably be straight, at an angle of about 50° to the vertical.

2. *Discussion of the Preceding Reflections.*—In the graphs 1 to 4, there are three influences at work on the resonator. In the first place the direct ray from the pipe is equivalent to a reflection from the table, a locus of reënforcement. A half wave-length has been lost. Then there are the two reflections, also with loss of $\lambda/2$, from the board normal to y and the wall W at $y = -130$ cm. This may be neglected, because it is always kept relatively very distant in the plan of the experiments. We thus have to compound the ray from the image of the source with the direct ray. The computation would be cumbersome; but the construction in the graph itself is not so and adequately correct. In figure 1, if the promi-

nent maxima or minima encountered during the march of the reflector in y, be taken, a circle is to be drawn with a center at R and a radius RP. In case of the first minimum at $y = 40$ cm., the image is I at $z = 40$, $y = 80$. Hence $IC = 27$ cm. is the path difference from the minimum at $y = 40$ cm. which is placed a little too high. The following data so measured, may be given as examples.

Minimum	$y =$ 40 cm.,	$\lambda/2 = 27$cm.	$y =$ 110 cm.,	$7\lambda/2 =$ 179 cm.
	70 cm.,	$3\lambda/2 = 82$		$9\lambda/2 =$ 220
	93 cm.,	$5\lambda/2 = 126$		
Maxima	$y =$ 53 cm.,	$1\lambda = 50$ cm.	$y =$ 130 cm.,	$4\lambda =$ 198 cm.
	83 cm.,	$2\lambda = 105$	155 cm.,	$5\lambda = 248$
	103 cm.,	$3\lambda = 148$	25 cm.,	$6\lambda = 4$

These results in their entirety are convincing, being quite as accurate as the location of maxima and minima admits. The minima of figure 2 treated by the same method give, for instance,

Minimum	$y =$ 25 cm.,	$\lambda/2 = 24$ cm.	$y =$ 103 cm.,	$7\lambda/2 = 170$ cm.
	50 cm.,	$3\lambda/2 = 68$	126 cm.,	$9\lambda/2 = 215$
	80 cm.,	$5\lambda/2 = 125$	etc.	

with similar data for the maxima.

* Advance note from a report to the Carnegie Institution, Washington, D. C.

NOTE ON THE PROBLEM OF GREAT STELLAR DISTANCES

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Communicated by Edwin B. Wilson, February 27, 1922

Through a study of the faint variable stars in the Small Magellanic Cloud it has been possible to check the photometric methods of measuring the parallaxes of globular clusters.

The great distances now ascribed to the globular systems require that the cluster type variables, with periods less than one day, be of high luminosity. The high absolute brightness of typical Cepheids, with periods longer than one day, is commonly accepted; but the faintness of the galactic variables of the cluster type, and their peculiarly large radial velocities, suggest that studies of their proper motions and trigonometric parallaxes will contribute little in the near future to the problem of their average absolute magnitude. It is now found, however, that the faintest variables in the Small Magellanic Cloud are Cepheids of the cluster type, and they are but one magnitude fainter than the typical Cepheids with periods of three days.

In 1904 and 1905 Miss Leavitt discovered nearly a thousand variable stars in the Small Magellanic Cloud from photographs made at Harvard's observing station at Arequipa, Peru. In addition, she found more than eight hundred variables in the Large Magellanic Cloud. Miss Leavitt

undertook an investigation of the periods and light curves of these stars, but for various reasons had not progressed far at the time of her death, beyond the work reported in *Harvard Circular* 173, 1912. In that paper she gives the periods of twenty-five stars, notes that they are all of the Cepheid type, and calls attention to a remarkable relation connecting the brightness of the variables and the length of their periods. The logarithm of the period was found to increase about 0.48 for each increase of one magnitude in brightness. The longest period she found was 127 days for one of the brightest stars in the Cloud; the shortest period was 1.25 days for a variable nearly four magnitudes fainter.

In the globular clusters are found large numbers of Cepheid variable stars with periods shorter than one day, and only a few with longer periods. In my work on cluster distances I used the relation given by Miss Leavitt, first putting it on an absolute luminosity basis (following Hertzsprung), and then extending it to these cluster type Cepheids. It soon appeared that the linear formula given by Miss Leavitt did not hold, and an empirical curve was derived and used for obtaining distances of Cepheids in the clusters and throughout the galactic system in general.

Some objections have been raised to my extension of the period-luminosity curve, which identifies cluster type variables as giants rather than dwarfs. The proper motions of two of these variables are large, and many of them are in high galactic latitude. Both properties are usually, but not always, associated with low absolute magnitude.

The adopted high luminosity is strongly supported, however, by the comparable brightness in clusters of the short period Cepheids and the B type stars, as well as by the direct comparison with the few long period Cepheids in Messier 5, Omega Centauri, and other globular clusters. Spectrophotometric analyses of the cluster stars afford additional support. A direct extension of the observed period-luminosity curve to cluster type variables in the Small Magellanic Cloud itself would appear to be especially decisive.

Most of the known variable stars in the Cloud have median magnitudes corresponding to periods between two and five days. A few are fainter. About twenty-five of these faint variables have been investigated during the past year on a special series of long-exposure photographs made for

Variable	Period *d.*	Median	Variable	Period *d.*	Median
1408	0.715	16.0	1823	0.795	16.0
1420	0.67*	16.4	1928	0.608	15.8
1634	0.81	15.6	1943	0.81*	16.2
1655	0.444:	16.2	1964	0.687*	16.1
1731	0.65	16.2	2002	0.702	16.4
1739	0.63*	16.4	3610	0.6	16.2
1741	0.400:	16.2			

the purpose at Arequipa. Further observation is necessary before definitive results can be given, but for the preceding variables provisional periods and median magnitudes have been obtained.

The numbers, except the last, are those published by Miss Leavitt. Colons indicate periods not well determined. For stars marked with an asterisk a multiple of the period given above satisfies the observations nearly or quite as well.

The light curves are in all cases of the typical cluster form. From the plates now available, the elements of variation could not be determined for some of the other faint stars, but for none do the observations demand a long period.

The median magnitudes are derived from published values of maximum and minimum, except that for No. 1741 which is a revised unpublished value derived by Miss Leavitt, and that for No. 3610—a hitherto unannounced variable. The medians are based on a provisional and apparently homogeneous system of magnitudes for the comparison stars. The standardization of these magnitudes is under way and will permit a new evaluation of the medians for the variable stars and a redetermination of the distance of the Small Magellanic Cloud.

The mean of the thirteen periods is 15.4 hours. The mean of the thirteen median magnitudes is 16.13 ± 0.05, in close agreement with the median magnitude, 16.2, predicted from the extended period-luminosity curve for variables of this period. We are led to the conclusion, even after making ample allowance for the provisional nature of the magnitude scale and for the uncertainties that affect some of the periods, that cluster type variables are giant stars in the Small Magellanic Cloud, as previously found for certain globular clusters. Naturally we infer that they are giants stars wherever they occur.

THE VITAL INDEX OF THE POPULATION OF ENGLAND AND WALES, 1838–1920[1]

BY RAYMOND PEARL AND MAGDALEN H. BURGER

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Communicated, February 14, 1922

The senior author[2] has suggested the term "vital index" as a convenient designation of the function

$$V = \frac{100\ Births}{Deaths}$$

which measures more effectively than any other demographic function yet devised the essential biological fitness of a population, in the sense of organic evolution. For a population V is the direct measure of survival

value. If $V > 100$ the population is not only surviving but growing, whereas if $V < 100$ the population is starting on the road to elimination, and must eventually disappear unless the vital index gets above 100. A detailed presentation of the facts regarding the vital index of certain elements of the American population has lately been published.[3]

In examining the effect of the Great War upon the growth of population one cannot fail to be impressed by the fact that the return of the birth/death ratio (vital index) to a normal or even super-normal value after the marked dislocation produced by the war, and at its end the superimposed influenza pandemic, was extraordinarily rapid. Indeed the senior author[4] was led to remark, after examining the matter in the chief belligerent countries, that "Altogether, these examples, which include the effects of the most destructive war known to modern man, and the most devastating epidemic

TABLE 1

RATIO V, OF 100 BIRTHS TO DEATHS IN ENGLAND AND WALES IN EACH QUARTER OF THE YEARS 1838 TO 1920, INCLUSIVE

	QUARTERS ENDED					QUARTERS ENDED					QUARTERS ENDED				
YEAR	MAR.	JUNE	SEPT.	DEC.	YEAR	MAR.	JUNE	SEPT.	DEC.	YEAR	MAR.	JUNE	SEPT.	DEC.
1838	116	134	157	140	1866	142	150	153	158	1894	155	186	201	180
1839	138	146	157	141	1867	145	178	176	158	1895	141	179	174	161
1840	134	143	148	135	1868	166	184*	147	160	1896	163	185	181	168
1841	135	151	164	150	1869	153	159	166	149	1897	164	179	169	170
1842	141	155	150	148	1870	143	168	155	151	1898	154	183	166	169
1843	144	150	167	150	1871	151	167	159	144	1899	157	182	151	152
1844	142	160	163	143	1872	155	173	170	176	1900	132	166	175	167
1845	137	153	177	163	1873	163	174	178	160	1901	158	182	169	167
1846	162	166	136	128	1874	157	176	169	150	1902	153	179	209	168
1847	122	130	136	123	1875	132	164	174	159	1903	170	195	207	169
1848	116	150	160	144	1976	162	179	180	177	1904	157	192	178	165
1849	145	150	100	139	1877	171	170	195	177	1905	160	192	195	172
1850	147	168	171	159	1878	158	177	172	155	1906	169	189	181	166
1851	149	160	165	150	1879	145	167	210	160	1907	143	187	216	161
1852	152	158	151	152	1780	152	186	167	165	1908	153	201	208	170
1853	137	147	160	140	1881	163	186	196	177	1909	143	189	218	172
1854	144	168	136	133	1882	159	182	186	163	1910	163	203	218	168
1855	123	155	177	153	1883	158	171	189	167	1911	156	188	158	169
1856	164	173	173	164	1884	170	181	167	165	1912	154	188	216	170
1857	157	170	160	146	1885	157	168	191	173	1913	148	187	199	172
1858	136	158	160	133	1886	147	184	179	168	1914	149	182	197	159
1859	144	166	161	155	1887	153	176	178	163	1915	125	154	180	134
1860	149	157	190	158	1888	149	179	199	169	1916	136	165	193	134
1861	143	172	170	159	1889	158	183	180	165	1917	109	136	172	136
1862	149	173	187	150	1890	136	171	180	139	1918	118	144	147	67
1863	145	160	154	155	1891	145	139	192	156	1919	75	137	197	195
1864	135	161	161	144	1892	120	176	197	168	1920	197	208	245	180
1865	138	167	160	148	1893	160	177	160	147					

TABLE 2

GROUPED DATA FOR VITAL INDEX AND CRUDE BIRTH RATE

PERIOD	BIRTH	DEATHS	100 BIRTHS / DEATHS	GENERAL BIRTH RATE PER 1000
1838–1839	956,361	681,744	140.28	31.0
1840–1844	2,600,288	1,756,431	148.04	32.2
1845–1849	2,797,329	2,003,657	139.61	32.6
1850–1854	3,080,095	2,030,528	151.69	33.9
1855–1859	3,300,929	2,126,461	155.23	34.3
1860–1864	3,560,830	2,263,769	157.30	34.9
1865–1869	3,830,527	2,438,121	157.11	35.3
1870–1874	4,100,856	2,541,625	161.35	35.5
1875–1879	4,399,070	2,623,391	167.69	35.6
1880–1884	4,451,771	2,591,038	171.81	33.8
1885–1889	4,450,173	2,620,108	169.85	32.0
1890–1894	4,486,912	2,778,642	161.48	30.5
1895–1899	4,611,116	2,771,151	166.40	29.6
1900–1904	4,691,038	2,739,365	171.25	28.5
1905–1909	4,637,271	2,613,992	177.40	26.7
1910–1914	4,411,823	2,519,713	175.09	24.2
1915–1919	3,623,894	2,685,456	134.95	20.8
1920	957,994	466,213	205.48	25.5

since the Middle Ages, furnish a substantial demonstration of the fact that population growth is a highly self-regulated biological phenomenon. Those persons who see in war and pestilence any absolute solution of the world problem of population, as postulated by Malthus, are optimists indeed. As a matter of fact, all history definitely tells us, and recent history fairly shouts in its emphasis, that such events make the merest ephemeral flicker in the steady onward march of population growth."

In view of the fact that this function V has so quickly returned to normal, or approximately normal, values after the recent disturbances it is a matter of considerable interest and significance to the student of problems of public health in the broad sense, of population growth, and of human ecology, to see how precisely the self-regulatory mechanism of populations operates over long periods of time. To this end we have examined all the existing continuous statistics of England and Wales, and desire to present the results in this paper.

Table I gives the vital index (100 births/deaths) for the population of England and Wales, in each quarters of the years 1838 to 1920 inclusive. The data from which the computations were made were taken from official sources.[5]

The immediately striking feature of this table is the extreme constancy of the values as one runs down the columns. In order to appreciate this fact fully, however, it is necessary to resort to graphical presentation. As the data of table I are too detailed to make such an examination of

the trend feasible within the space of a single printed page, the data have been combined in quinquennial periods in table 2. In this table are given, for each 5 year period, the total births, total deaths, vital index, and crude birth rate. The latter figures have been calculated directly from the birth and population figures and differ slightly (but in no wise significantly) from the crude birth rates given in Annual Reports of the Registrar-General, chiefly because of the fact that we have used a different quinquennial grouping.

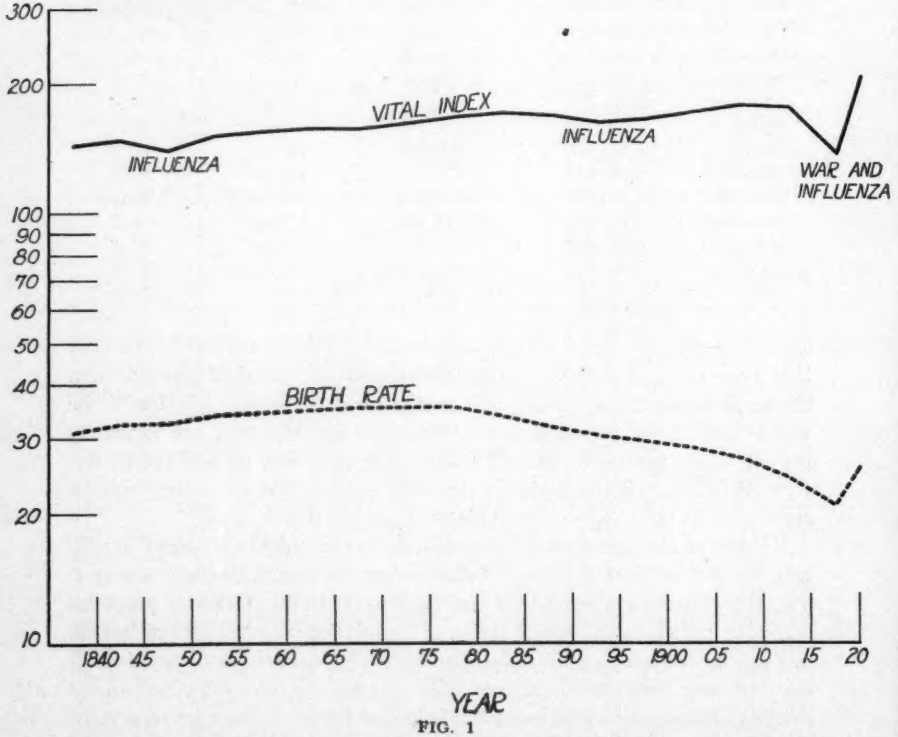

FIG. 1

Trend of vital index (100 births/deaths) and crude birth rate in England and Wales, 1838–1920, inclusive

The vital index and birth rate data from table 2 are shown graphically in figure 1.

The diagram is plotted on an arithlog grid, in order that the slopes of the lines may be strictly comparable.[6]

It is at once apparent that *the ratio of births to deaths in England and Wales had a slow but extremely even and steady increase from 1838 to 1914.* This steady progress was interrupted to a degree sufficient to be apparent

upon only two occasions during the three quarters of a century. These were in 1847–1849 and 1890–1892. These fluctuations, which only slightly affected the even upward trend of the curve were due to the influenza pandemics of 1847–8 and 1890–1891. The broad result is perfectly clear and outstanding. The population of England and Wales is today biologically fitter and possessed of greater purely biological survival value as a whole population than it was three quarters of a century ago. Whether it is a mentally, morally or anthropmetrically fitter population does not now concern us. We are dealing at the moment solely with the fact that, taking the people of England and Wales as a whole, slightly over 2 babies were born for every death in a year in 1920, as against 1.4 babies per death in a year in 1838–1839. The question of a differential fertility among different social classes we shall discuss farther on in the paper.

Now this result, which is beyond any question a fact, will strike anyone informed as to the sociological and eugenical literature of the last two decades as curiously at variance with the pessimistic tenor of that literature, taken as a whole. It has been pronounced from high places that the general trend of the British people was biologically downwards, that they were in fact becoming biologically dangerously near to a decadent race. Abundant quotations in support of this contention could be cited, were space available and were it necessary. This gloomy view has had its foundation solely upon the fact that, since the quinquennium 1875–1880, the birth rate in England and Wales has been falling rather rapidly, as is clearly shown in figure 1. This fact has been brought out by Elderton[7] in great detail.

But from a purely biological view-point, what matters a falling birth rate if the death rate falls even more rapidly, so that the net survivorship at any instant of time is constantly getting higher? To this it will, of course, be answered at once by those who view with alarm the declining birth rate that the real crux of the matter is in the differential change in fertility. Nowadays the "best" people do not produce their due share of progeny, while the "worse" people over-produce. If one accepts the man-made definitions of "best" and "worst" this is plainly true for some civilizations at least. In the American population, however, as one of us[3] has recently shown, the element probably least desirable racially, the negro, has the lowest survival value as a group (vital index generally less than 100). But is old Dame Nature ever really interested in any test of "best" and "worst" *except survival?* There is no evidence as yet that she is. Measured by this test the population as a whole of England and Wales is today biologically more vigorous than it was in 1838. This is a plain *fact.* Whether this fact is, sociologically or eugenically considered, a bad thing or a good thing, is something more than a purely biological problem, and as such we are content to leave it to the discussion of others.

What seems to us of considerable interest biologically, is the extraordinary perfection, demonstrated in the figures here presented, of the self-regulation of population growth. Disturbances of the course of the birth rate have been compensated for, in the population and period under review, with the greatest nicety. It will be of interest to examine the facts for other populations, and this we hope later to do.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 50.

[2] R. Pearl, "The Effect of the War on the Chief Factors of Population Change," *Science* (N. S.), **51**, 1920 (553–556).

[3] Pearl, "The Vitality of the Peoples of America," *Amer. J. Hyg.*, **1**, 1921, pp. 592–674.

[4] Pearl, "A Further Note on War and Population," *Science* (N. S.), **53**, 1921, pp. 120–121.

[5] *Quarterly Return of Marriages, Births, and Deaths Registered in England and Wales, etc.* Published by Authority of the Registrar-General. London. H. M. Stationery Office. *Passim.* Also corresponding Weekly Return, and Annual Reports.

[6] "For a Discussion of the Significance and Advantages of Plotting Trend Lines on a Logarithmic Scale," cf. Fisher, I. "The 'ratio' Chart for Plotting Statistics," *Q. P. Am. Stat. Ass.*, **15**, 1917, pp. 577–601, and Field, J. A., "Some Advantages of the Logarithmic Scale in Statistical Diagrams," *J. Pol. Econ.*, **25**, 1917, pp. 805–841.

[7] E. M. Elderton, "Report on the English Birth Rate. Part I. England North of the Humber." *Eugenics Lab. Mem.*, **19** and **20**, Cambridge Univ. Press, 1914, pp. viii and 246.

SEASONAL FLUCTUATIONS OF THE VITAL INDEX OF A POPULATION[1]

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In earlier papers the writer[2] has discussed various aspects of the birth-death ratio, or vital index of a population, including age and sex specific vital indices, racial and secular fluctuations, etc. Table 1 of the preceding paper by Pearl and Burger[3] gives the necessary data for finding, for one population, that of England and Wales, the course of the seasonal fluctuation of this important demographic constant. It is the purpose of this note to set forth biometrically the facts on this point, from the basic data mentioned.

Table 1 gives the frequency distributions for variation in the value of the vital index of the population of England and Wales in each of the four quarters of the year, ending, respectively, on March 31, June 30, September 30, and December 31, during the years 1838 to 1920, inclusive. The source of the data is given in Pearl and Burger already cited.

The significant biometric constants from table 1 are presented in table 2. It is apparent that there are a few highly aberrant observations in

TABLE 1

FREQUENCY DISTRIBUTION FOR VARIATION IN THE VITAL INDEX IN THE FOUR QUARTERS OF THE YEAR

VITAL INDEX	QUARTERS ENDING			
	MARCH 31	JUNE 30	SEPT. 30	DEC. 31
65– 69	—	—	—	—
70– 74	—	—	—	—
75– 79	1	—	—	—
80– 84	—	—	—	—
85– 89	—	—	—	—
90– 94	—	—	—	—
95– 99	—	—	—	—
100–104	—	—	1	—
105–109	1	—	—	—
110–114	—	—	—	—
115–119	3	—	—	—
120–124	3	—	—	1
125–129	1	—	—	1
130–134	3	2	—	4
135–139	9	3	3	4
140–144	11	2	—	7
145–149	13	2	3	5
150–154	9	7	5	9
155–159	13	6	5	11
160–164	10	5	10	9
165–169	2	11	9	17
170–174	3	9	8	7
175–179	—	11	8	4
180–184	—	9	6	2
185–189	—	10	3	—
190–194	—	2	4	—
195–199	1	1	8	1
200–204	—	2	1	—
205–209	—	1	3	—
210–214	—	—	1	—
215–219	—	—	4	—
245–249	—	—	1	—
Total	83	83	83	83

TABLE 2

BIOMETRIC CONSTANTS DEDUCED FROM TABLE 1

QUARTER ENDING	MEAN	STANDARD DEVIATION
March 31	147.06 ± 1.21	16.32 ± .85
June 30	169.89 ± 1.21	16.35 ± .86
September 30	175.43 ± 1.66	22.39 ± 1.17
December 31	156.46 ± 1.27	17.14 ± .90

the series, but it has not been thought best to discard or adjust them in this first treatment of the subject. Consequently it is to be understood that all the values given in table 1 were used in computing the values in table 2.

From these data the following points are to be noted:

1. The vital index has the lowest mean value in the quarter ending on March 31, the winter quarter. In that period the birth incidence is relatively low and the death incidence relatively high.

2. Next in value to this, but standing 9.40 ± 1.75 points above it, is the mean vital index for the autumn quarter ending December 31. The difference being 5.4 times its probable error, may be regarded as significant.

3. The spring quarter, ending June 30, shows the next higher mean, being 13.43 ± 1.75 points above the winter quarter.

4. The highest value of the index falls in the summer quarter, when births are most frequent and deaths least so. The mean value, however, lies only 5.54 ± 2.05 above that for the spring quarter, a difference which cannot be regarded as significant.

5. In variability of the vital index, the first two and the last quarters of the year, all exhibit significantly the same status. The vital index is distinctively more variable in the summer quarter, the difference in standard deviations when this quarter is compared with that ending June 30, amounting to 6.04 ± 1.45. This may be regarded as significant, being 4.2 times its probable error.

It thus appears that the extremely close compensatory relation between birth rate and death rate, which Pearl and Burger[3] have shown to hold in annual figures, does not obtain within the single year. Instead there is a well-marked statistically significant intra-annual, or seasonal fluctuation of the birth-death ratio.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 51.

[2] Cf. particularly Pearl, R., "The Vitality of the Peoples of America," *Amer. J. Hyg.*, **1**, 1921 (592–674).

[3] Pearl, R. and Burger, M. H., "The Vital Index of the Population of England and Wales, 1838–1920," *Proc. Nat. Acad. Sci.*, **8**, 1922, pp. 71–76.

A SOLUTION OF THE LINEAR MATRIX EQUATION BY DOUBLE MULTIPLICATION

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1. *Methods of Sylvester and of Maclagan Wedderburn.*—A matrix A of order N is defined as a binary assemblage of N^2 elements a_{ik}, where either subscript may have any value from 1 to N. We add two matrices by adding corresponding elements. We multiply by the rule

$$(AB)_{ik} = \Sigma a_{is} b_{sk}; (s = 1, 2, \ldots, N). \qquad (1)$$

If $A_1, A_2, \ldots, A_h$ and $B_1, B_2, \ldots, B_h$ and C are all known matrices, while x is a required matrix, the linear matrix equation may be written

$$\theta(x) = A_1xB_1 + A_2xB_2 + \ldots + A_hxB_h = C. \qquad (2)$$

In 1884 Sylvester attacked this equation with characteristic energy.[1] He pointed out that, since each element of the matrix $\theta(x)$ is a linear function of the elements of x—the equation (2) being thus equivalent to N^2 linear equations of ordinary algebra—it follows θ has the properties of a matrix of order N^2 and satisfies a Hamilton-Cayley equation

$$\theta^n - m_1\theta^{n-1} + m_2\theta^{n-2} - \ldots + (-1)^n m_n I = 0 \qquad (3)$$

where $n = N^2$ and where I denotes the matrix of order N^2 whose elements I_{ik} are unity when subscripts are equal, otherwise zero. If we know the coefficients $m_1 \ldots m_n$ we can solve (2), as is evident on multiplying (2) and (3) through by θ^{-1}. We can find all the coefficients if we know the rule for finding m_n which is the determinant of the matrix θ; for the left side of (3) may be found by forming the determinant of $gI - \theta$ and afterwards putting θ in place of g. *To Sylvester, solution of the equation (2) was equivalent to finding, in terms of the matrices $A_1 \ldots A_h$ and $B_1 \ldots B_h$ the determinant of order N^2 belonging to the operation θ.*

His general method consists in setting up the two matrices of order N and of extent (étendue) h

$$x_1A_1 + x_2A_2 + \ldots + x_hA_h,\ x_1B_1 + x_2B_2 + \ldots + x_hB_h$$

where x_1, x_2, etc. are scalar variables. By forming the determinants of these two matrices we have two quantics in the variables. He shows that the required determinant is a *contrariant* called the *nivellant* of these two quantics, and is an integral but not necessarily a rational function of the coefficients of the quantics. These coefficients are regarded as known quantities. The formation of the contrariant is in general difficult. He works out the solution for several special cases. With regard to the general case he remarks[2]

"Pour présenter l'expression générale de ce déterminant pour une matrice d'un ordre et d'une étendue quelconques, c'est-à-dire pour résoudre l'équation linéaire en matrices dans toute sa généralité, il faudrait avoir une connaissance des propriétés des formes qui va beaucoup au delà des limites des facultés humaines, telles qu'elles se sont manifestées jusqu'au temps actuel, et qui, dans mon jugement, ne peut appartenir qu'à l'intelligence suprême."

But the indeterminate product of Gibbs was at that time unborn, or at least very young.

In 1904 Maclagan Wedderburn[3] gave a solution of (2) by infinite series. His method consists essentially in selecting one of the known matrices, as A_j, multiplying *by* A_j^{-1} and *into* B_j^{-1} giving (2) in the form

$$x + \psi(x) = A_j^{-1} C B_j^{-1} = C'$$

where ψ is similar in form to θ but contains h-1 terms. Then

$$x = (1 + \psi)^{-1} C' = (1 - \psi + \psi^2 \ldots)C'.$$

The choice of the term A_ixB_j to be freed from matrix coefficients affects the convergence of the solution.

2. *Transformation from Matrices to Double Dyadics.*—By Gibbs' double multiplication we may set up explicit formulas for the coefficients $m_1 \ldots m_n$ of the Hamilton-Cayley equation (3). It is known that any matrix A is equivalent to a dyadic $\Sigma\mathbf{aa}'$ where $\mathbf{a}$ and $\mathbf{a}'$ are vectors. Any matrix term of the form AxB is equivalent to a sum of terms of the form $\mathbf{aa}'.\mathbf{X}.\mathbf{bb}'$ where $\mathbf{X}$ is the dyadic equivalent to x. We now introduce the double dot product defined by the rule $\mathbf{ab}:\mathbf{xy} = \mathbf{a}.\mathbf{xb}.\mathbf{y}$, and may write

$$\mathbf{aa}'.\mathbf{X}.\mathbf{bb}' = \mathbf{ab}'\mathbf{a}'\mathbf{b}:\mathbf{X}. \tag{4}$$

Any term AxB is therefore equivalent to a sum of terms of the form $\mathbf{MN}:\mathbf{X}$ where $\mathbf{M}$ and $\mathbf{N}$ are dyadics, and the linear matrix equation (2) is equivalent to

$$\varphi:\mathbf{X} = \Sigma\mathbf{MN}:\mathbf{X} = \mathbf{C} \tag{5}$$

where φ is a dyadic whose antecedents and consequents are dyadics, say a *double dyadic*.

3. *Dyadics Treated Like Vectors in Space of Higher Dimensions.*—Adopting unit vectors $\mathbf{e}_1 \ldots \mathbf{e}_n$ such that $\mathbf{e}_i.\mathbf{e}_k$ is unity when subscripts are equal, otherwise zero, any dyadic $\mathbf{M}$ may be expanded in terms of the dyads $\mathbf{e}_i\mathbf{e}_k$; and if we agree on some definite order, no matter what, among these dyads we may set $\mathbf{E}_r = \mathbf{e}_i\mathbf{e}_k$, *lettering r run from 1 to N^2 as i and k each run from 1 to N.* With $n = N^2$, any dyadic $\mathbf{M}_t$ may be expanded

$$\mathbf{M}_t = \mathbf{E}_1 m_{t1} + \mathbf{E}_2 m_{t2} + \ldots + \mathbf{E}_n m_{tn} \tag{6}$$

where m_{t1} etc. are scalars. The double dyadic φ may be expanded as

$$\varphi = \mathbf{E}_1\mathbf{M}_1 + \mathbf{E}_2\mathbf{M}_2 + \ldots + \mathbf{E}_n\mathbf{M}_n. \tag{7}$$

Now $\mathbf{e}_i\mathbf{e}_k:\mathbf{e}_r\mathbf{e}_s$ is unity when *both $i = r$ and $k = s$*, otherwise zero. This is the same assaying $\mathbf{E}_p:\mathbf{E}_q$ is unity when subscripts are equal, otherwise zero. It is immediately evident that the double dyadic φ will behave with reference to double dot product as would an ordinary dyadic in n- dimensional space with reference to ordinary dot product, and equally evident that the whole of the usual matrix theory will apply to φ with no essential change.

4. *Formation of the Hamilton-Cayley Equation.*—Since $\varphi:\mathbf{X} = \theta(x)$ the Hamilton-Cayley equation for φ will be the same as that for θ. It remains to show how to calculate the coefficients $m_1 \ldots m_n$ of this equation. Let $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_p$ and $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_p$ be any dyadics:

Definition.—The scalar $((\mathbf{M}_1\mathbf{N}_1, \mathbf{M}_2\mathbf{N}_2, \ldots, \mathbf{M}_p\mathbf{N}_p))_S$ is defined to be the sum of terms computed as follows: the leading term is the product of double dot products $\mathbf{M}_1:\mathbf{N}_1\mathbf{M}_2:\mathbf{N}_2 \ldots\ldots \mathbf{M}_p:\mathbf{N}_p$; the other terms are of like form and are obtained from the leading term by keeping the antecedents $\mathbf{M}_1 \ldots . \mathbf{M}_p$ fixed in position while the consequents $\mathbf{N}_1 \ldots \mathbf{N}_p$ are permuted in all ways. The sign of any term is plus or minus according as the number of simple interchanges needed to form that term is even or odd.

Thus if $p = 2$ we shall have

$$((\mathbf{M}_1\mathbf{N}_1,\ \mathbf{M}_2\mathbf{N}_2))_S = \mathbf{M}_1:\mathbf{N}_1\mathbf{M}_2:\mathbf{N}_2 - \mathbf{M}_1:\mathbf{N}_2\mathbf{M}_2:\mathbf{N}_1.$$

If $\varphi_1, \varphi_2, \ldots, \varphi_p$ are double dyadics defined as in (7) the scalar $((\varphi_1, \varphi_2, \ldots, \varphi_p))_S$ is defined to be the result of expanding each φ as in (7), multiplying out term by term, and adding the scalars of all the terms of the product.

If each $\mathbf{M}$ be expanded as in (6) the scalar elements of every φ will form a square matrix m_{ts} of order N^2 or n, *so that each* φ *depends on* N^4 *scalar elements*. The scalar $((\varphi_1, \varphi_2, \ldots, \varphi_p))_S$ is the sum of cubic determinants of order p whose main diagonals lie on the main diagonals of the respective matrices m_{ts}; the p matrices m_{ts} are to be regarded as p nonsignant layers of a cubic matrix.[4]

We may now suppose these p double dyadics to be all equal. The scalar just written becomes $((\varphi, \varphi, \ldots \text{ to } p \text{ factors}))_S$ and may be abbreviated $((\varphi^p))_S$.

Theorem.—The scalar $((\varphi^p))_S = p!m_p$ where m_p is the coefficient of φ^{n-p} in the Hamilton-Cayley equation for φ.

Proof. Let φ be expanded as in (7). The terms of the indeterminate product $(\varphi, \varphi, \ldots \text{ to } p \text{ factors})$ are of the form

$$(\mathbf{E}_i\mathbf{M}_i, \mathbf{E}_j\mathbf{M}_j, \mathbf{E}_k\mathbf{M}_k, \ldots \mathbf{E}_r\mathbf{M}_r) \qquad (8)$$

where any subscript will have any value from 1 to n. Since $\mathbf{E}_s:\mathbf{M}_t = m_{ts}$ the scalar of the expression (8) has for its leading term $m_{ii}m_{jj}m_{kk}\ldots m_{rr}$ and, by the definition, the scalar is the determinant of order p of which this is the leading term, a minor of the determinant of φ. Thus the scalars of all expressions (8) vanish except when the subscripts $i, j, k, \ldots, r$ are all different. Each choice of subscripts $i, j, k, \ldots, r$ will occur $p!$ times. The scalar $((\varphi^p))_S$ is therefore $p!$ times the sum of minors of order p taken along the main diagonal of the determinant of φ. By the usual matrix theory then $((\varphi^p))_S = p!m_p$ as was to be proved.[5]

5. *Transformation Back to Matrices.*—It has thus been shown how the invariants $m_1 \ldots m_n$ of the Hamilton-Cayley equation for φ or θ may be calculated by products which are closely analogous with Gibbs' products of ordinary dyadics. To complete the solution we need to see how these invariants may be formed directly from the given matrices $A_1 \ldots A_h$ and $B_1 \ldots B_h$ without the need of first forming φ by the rule (4).

Consider first m_1, the coefficient of φ^{n-1}. If $\varphi = \Sigma\mathbf{MN}$ we have $m_1 = \Sigma\mathbf{M}:\mathbf{N} = \Sigma\mathbf{ab}':\mathbf{a}'\mathbf{b} = \Sigma\mathbf{a}.\mathbf{a}'\mathbf{b}.\mathbf{b}' = \Sigma A_S B_S$ where A_S and B_S are the scalars of the dyadics corresponding to A and B, or, in matrix language, they are the respective sums of the elements of the main diagonals. Thus

$$m_1 = A_{1S}B_{1S} + A_{2S}B_{2S} + \ldots + A_{hS}B_{hS}. \qquad (9)$$

For m_2 we have, by the distributive character of all the steps involved in the process,

$$\begin{aligned} 2m_2 &= ((\varphi, \varphi))_S = \Sigma((\mathbf{M}_i\mathbf{N}_i, \mathbf{M}_k\mathbf{N}_k))_S \\ &= \Sigma((\mathbf{a}_i\mathbf{b}'_i\mathbf{a}'_i\mathbf{b}_i, \mathbf{a}_k\mathbf{b}'_k\mathbf{a}'_k\mathbf{b}_k))_S \\ &= \Sigma[(\mathbf{a}_i\mathbf{b}'_i:\mathbf{a}'_i\mathbf{b}_i)\ (\mathbf{a}_k\mathbf{b}'_k:\mathbf{a}'_k\mathbf{b}_k) - (\mathbf{a}_i\mathbf{b}'_i:\mathbf{a}'_k\mathbf{b}_k)\ (\mathbf{a}_k\mathbf{b}'_k:\mathbf{a}'_i\mathbf{b}_i)] \\ &= \Sigma[\mathbf{a}_i.\mathbf{a}'_i\mathbf{b}_i:\mathbf{b}'_i\mathbf{a}_k.\mathbf{a}'_k\mathbf{b}_k.\mathbf{b}'_k - \mathbf{a}_i.\mathbf{a}'_k\mathbf{a}_k.\mathbf{a}'_i\mathbf{b}_k.\mathbf{b}'_i\mathbf{b}_i.\mathbf{b}'_k] \\ &= \Sigma[A_{iS}B_{iS}A_{kS}B_{kS} - (A_iA_k)_S(B_kB_i)_S] \qquad (10) \end{aligned}$$

where, as before, A_{iS} is the sum of elements of the main diagonal of A_i, etc. and $(A_iA_k)_S$ is the sum of elements of the main diagonal of the product of the two matrices A_i and A_k.

By similar transformation we may show that

$$\begin{aligned} 6m_3 &= \Sigma[A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS} - A_{iS}B_{iS}(A_jA_k)_S(B_kB_j)_S \\ &\quad - A_{jS}B_{jS}(A_kA_i)_S(B_iB_k)_S - A_{kS}B_{kS}(A_iA_j)_S(B_jB_i)_S \\ &\quad + (A_iA_jA_k)_S(B_kB_jB_i)_S + (A_kA_jA_i)_S(B_iB_jB_k)_S]. \qquad (11) \end{aligned}$$

In every case all subscripts run from 1 to h, *and terms where some or all of the subscripts are equal do not in general vanish.* For a matrix term AxB is not in general equivalent to a *single* dyadic term $\mathbf{MN:X}$.

The general rule for any m is now easy to see. The quantity $p!m_p$ may be written as a *summation of square bracket expressions, each square bracket enclosing p! terms. The leading term is*

$$\Sigma A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS} \ldots A_{rS}B_{rS}$$

where the p subscripts i, j, k, . . . r are those of the given matrix coefficients in (2) and may be either alike or different. The other terms are formed by first keeping the antecedents A fixed and making interchanges among the consequents B. A multiple interchange such that q consequents $B_aB_bB_c \ldots B_eB_fB_g$ *(whether alike or not) change places among themselves leads to a product of two factors as* $(A_cA_bA_e \ldots A_dA_fA_a)_S\ (B_aB_fB_d \ldots B_eB_bB_c)_S$ *where the order of consequents is the order they stand after the interchange, while the order of antecedents is the reverse of that of the consequents. Any antecedents or consequents which have not been moved yield the same scalars as in the leading term.*

These quantities m may be transformed in many ways.

6. *A Second Method of Solution.*—Another method of solving the equation (2) follows at once from the analogy between θ and an ordinary dyadic. If $\varphi = \Sigma\mathbf{NM}$ let $\varphi' = \Sigma\mathbf{NM}$ corresponding to θ', that is to $\Sigma A'xB'$. *Let* $C_1, C_2, \ldots C_{n-1}$ *be n-1 matrices or dyadics such that each is orthogonal to the given matrix C in the sense that* $C:C_i = 0$ *for* $i = 1, 2, \ldots, n-1$. *Form* $n-1$ *new matrices* $\theta'(C_i)$. *A matrix orthogonal to all these new matrices will in general be a solution of (2) aside from a scalar factor which may be found by direct substitution.* This scalar may also be found as a function of m_n and the matrices $C, C_1, C_2, \ldots C_{n-1}$.

This method is analogous with Hamilton's original method of solving the linear vector equation.[6]

7. *Conclusion.*—The analogy between a dyadic and a vector by virtue of the double dot product may be extended to polyadics and K-tuple dot

products, hence to dyadics whose antecedents and consequents are polyadics, say double polyadics. The analogy may also be applied to certain other products. Thus the scalar $((\mathbf{ab}, \mathbf{cd}))_S$ or $\mathbf{a.bc.d} - \mathbf{a.dc.b}$ may be regarded as a product of two dyadics $\mathbf{M}^*\mathbf{N}$ and dyadics found which are orthogonal with respect to star product, that is such that $\mathbf{M}_i{}^*\mathbf{N}_k$ is unity when subscripts are equal, otherwise zero. We may similarly have star products for double dyadics and double polyadics. We shall have Hamilton-Cayley equations with respect to star products, where the scalars m are formed by star product just as those of this paper were formed by double dot product. The present paper is intended as introduction to more detailed treatment of some of these phases of multiple algebra. They have a bearing on ordinary algebra by virtue of the fact that a polynomial of degree K in N variables corresponds to a symmetrical polyadic of order K in space of N dimensions.[7]

[1] In a series of *Paris* papers, *C. R. Acad. Sci.*, **98,** and **99** (1884) and elsewhere. The complete bibliography may be found in *Bull. Inter. Assoc. Promoting Study Quaternions and Allied Branches*, particularly for March, 1908.

[2] *Paris, C. R. Acad of Sci.*, **99,** p. 435.

[3] "Note on the Linear Matrix Equation," *Edinburgh Math. Soc. Proc.*, **22,** 1904 (49–53).

[4] The laws of signant and non-signant indices for p-way determinants were given by Lepine Hall Rice, *Amer. J. Math.*, **40,** No. 3, July, 1918.

[5] By forming the determinant of $\varphi - \lambda\mathrm{I}$.

[6] *Elements of Quaternions*, Vol. I, 2nd. Ed., p. 498.

[7] Binary quadrics as vectors in 3-space, with what I here call star orthogonality, have been studied in great detail by Emil Waelsch. See the bibliographies referred to in Note 1. His method is given in *Wien. Ber.*, **112,** 1903, p. 645, 1091, and 1533, and called "Binäre-analyse."